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Variational Principle for the $P(4)$ Affine Theory of Gravitation and Electromagnetism

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We propose a Lagrangian for the $P(4)$ theory of gravitation and electromagnetism which is a straightforward generalization of the Einstein Lagrangian. A constrained Palatini variation of this Lagrangian yields the geometrical Einstein-Maxwell affine field equations. We show that these results can be extended easily to include both electric and magnetic charges. Finally, we consider conservation laws arising from the invariance properties of the Lagrangian.

1. INTRODUCTION

The $P(4) = O(1, 3) \otimes \mathbb{R}^{4*}$ theory of gravitation and electromagnetism (Norris, 1985, 1991; Kheyfets and Norris, 1988) provides a complete geometrization of the source-free Einstein-Maxwell field equations by the identification of the Maxwell field strength tensor with the $\mathbb{R}^{4*}$ part of a $P(4)$ generalized affine connection (Kobayashi and Nomizu, 1963), thus placing it on the same geometrical level as the Riemannian linear connection. Until now, however, no variational principle has been found which yields the geometrical Einstein-Maxwell affine field equations. In this paper we present a Lagrangian whose variation produces these field equations. We shall also address certain questions which have remained concerning the translational gauge covariance of these equations.

We begin in Section 2 by giving a brief overview of the $P(4)$ theory. For a more complete description of the details the reader is referred to earlier work (Norris, 1985; Kheyfets and Norris, 1988). The bundle structure is described with particular emphasis on how basic quantities transform under translational gauge changes. We use the translational degrees of freedom to

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model the 4-momentum spaces of charged particles as four-dimensional affine spaces and are led to the geometrization of charged particle trajectories as affine 4-momentum geodesics. In the process we identify the $\mathbb{R}^{4*}$ part of the $P(4)$ generalized affine connection, in a particular translational gauge, with the negative of the Maxwell field strength tensor. It is this identification which enables us to geometrize the coupled Einstein–Maxwell equations.

The variational technique that we shall employ will be of the constrained Palatini type. In Section 3 we briefly review this widely applicable technique that in a sense yields a maximal amount of information, since it reveals the generalized forces of constraint. In Section 4 we introduce our Lagrangian and argue that it is a straightforward generalization of the Einstein Lagrangian of general relativity. We then show that the Lagrangian is invariant under translational gauge transformations, and, up to a total divergence, the Lagrangian is invariant under classical gauge transformations of the vector potential as well. Finally, the variation of the Lagrangian yields the source-free Einstein–Maxwell affine field equations while maintaining Riemannian linear geometry.

In Section 5 we consider the extension of our results to include electric and magnetic sources. The latter is accomplished by the introduction of a magnetic vector potential. In Section 6 we examine the conservation laws which arise from the invariance properties of the $P(4)$ Lagrangian. Section 7 includes a summary of our results and a discussion of their implications.

2. THE $P(4)$ THEORY OF ELECTROMAGNETISM AND GRAVITY

The geometrical arena of the $P(4)$ theory of gravitation and electromagnetism is the modified affine frame bundle, $\hat{A}M$ over a four-dimensional spacetime manifold M. Elements of $\hat{A}M$ are triples $(p, e_i, \hat{t})$, where $p \in M$, (e_i) is a linear frame at p, and $\hat{t}$ is an affine cotangent vector, the "origin" of the frame at p. This modification[3] is necessary because we wish to model the 4-momentum spaces of charged particles as affine spaces[4] and the 4-momentum is fundamentally a covector rather than a vector. The structure group of $\hat{A}M$ is the affine group $\hat{A}(4) = Gl(4) \otimes \mathbb{R}^{4*}$ with group multiplication

$$(A_1, \xi_1) \cdot (A_2, \xi_2) = (A_1 A_2, \xi_1 \cdot A_2 + \xi_2)$$

for all (A_1, ξ_1), $(A_2, \xi_2) \in \hat{A}(4)$ (Norris, 1991). Since $\hat{A}M$ is bundle isomorphic to AM, the standard affine frame bundle, we shall simplify the

[3]Ordinarily, the affine frame bundle AM is the set of triples $(p, e_i, \hat{t})$, where $\hat{t}$ is an affine tangent vector. The structure group of this bundle is $A(4) = Gl(4) \otimes \mathbb{R}^4$ (Kobayashi and Nomizu, 1963).

[4]An affine space (Dodson and Poston, 1977) is a triple (S, V, δ), where S is a set, V a vector space, and $\delta: S \times S \to V$ such that, for $\hat{x}, \hat{y}, \hat{z} \in S$, (1) $\delta(\hat{x}, \hat{y}) + \delta(\hat{y}, \hat{z}) = \delta(\hat{x}, \hat{z})$ and (2) for all $\hat{x} \in S$, the map $\delta_{\hat{x}}(\hat{y}) = \delta(\hat{y}, \hat{x})$ is a bijection.

terminology and notation by referring to $\hat{A}M$ as the affine frame bundle of M and denote it by AM. Moreover, we will denote by $P(4)$ the Poincaré subgroup $O(1,3)\otimes\mathbb{R}^{4*}$ of $\hat{A}(4)$.

AM is a principal fiber bundle over LM with standard fiber $\mathbb{R}^{4*}$. We shall refer to sections of AM over LM as translational gauges. A translational gauge can be thought of, therefore, as a choice of origin for local 4-momentum affine frames on M. It can be shown that translational gauges are in one-to-one correspondence with covector fields on M (Norris, 1991).

A generalized affine connection on AM can always be specified (Kobayashi and Nomizu, 1963) by a pair $(\Gamma, {}^{\hat{t}}K)$ on spacetime, where Γ is a linear connection and ${}^{\hat{t}}K$ is a covariant vector-valued one-form, where the left superscript indicates that K is represented in the $\hat{t}$ translational gauge. If the linear connection is Riemannian, then the pair is said to represent a $P(4)$ connection. Furthermore, the pair $(\{\cdot\}_g, {}^{\hat{t}}K)$ may be used to construct the pair $(R, {}^{\hat{t}}\Phi)$ where R is the Riemannian curvature and ${}^{\hat{t}}\Phi$ is a covariant vector-valued 2-form on spacetime, the affine or $\mathbb{R}^{4*}$ curvature. Its components are defined by[5]

$$ {}^{\hat{t}}\Phi_{ijk} \stackrel{\text{def}}{=} {}^{\hat{t}}K_{kj;i} - {}^{\hat{t}}K_{ki;j} \tag{2.1} $$

Under a translational gauge transformation, $\hat{t}\to\hat{x}=\hat{t}\oplus\vec{s}$, the $\mathbb{R}^{4*}$ connection transforms as[6]

$$ {}^{\hat{t}\oplus\vec{s}}K_{ij} = {}^{\hat{t}}K_{ij} + s_{i;j} \tag{2.2} $$

and therefore, under the same transformation, we have

$$ {}^{\hat{t}\oplus\vec{s}}\Phi_{ijk} = {}^{\hat{t}}\Phi_{ijk} - R_{ijk}{}^{l}s_l \tag{2.3} $$

If we define the contraction

$$ {}^{\hat{t}}\Phi_i \stackrel{\text{def}}{=} g^{jk}({}^{\hat{t}}\Phi_{ijk}) \tag{2.4} $$

then we obtain for it, from equation (2.3), the transformation law

$$ {}^{\hat{t}\oplus\vec{s}}\Phi_i = {}^{\hat{t}}\Phi_i - R_i^{\,j}s_j \tag{2.5} $$

Physically, we shall model the local 4-momentum spaces of classical charged particles as four-dimensional affine spaces (Norris, 1985). In such a space the observed 4-momentum must always be expressed relative to some *local zero of affine 4-momentum.* By this we mean that the observed 4-momentum is a vector ${}^{\hat{\sigma}}\vec{\pi}$, such that $\hat{\pi}=\hat{\sigma}\oplus{}^{\hat{\sigma}}\vec{\pi}$, where $\hat{\pi}$ is the affine 4-momentum and $\hat{\sigma}$ is the local zero (i.e., reference) of affine 4-momentum. We assume that there exists a translational gauge $\hat{0}$ such that, at a point

[5]For a linear connection with torsion ${}^{\hat{t}}\Phi_{ijk} \stackrel{\text{def}}{=} {}^{\hat{t}}K_{kj;i} - {}^{\hat{t}}K_{ki;j} + S_{ij}{}^{m}\,{}^{\hat{t}}K_{mk}$, where $S_{ij}{}^{k}=\Gamma_{[ij]}{}^{k}$.
[6]The notation $\hat{x}=\hat{y}\oplus\vec{s}$ means that $\vec{s}=\delta(\hat{y},\hat{x})=\delta_{\hat{x}}(\hat{y})$.

along its trajectory in a nonzero electromagnetic field, the observed 4-momentum of the charged particle is the same as that of an instantaneously comoving and freely falling uncharged particle. In other words, $\hat{\pi} = \hat{0} \oplus \vec{u}$, where $\vec{u}$ is the 4-momentum per unit mass of the uncharged reference particle. We call $\hat{0}$ the *zero translation gauge* (Norris, 1985).

In order to transport the local zero of 4-momentum, defined at any single event in spacetime, to other events in spacetime, we must utilize an *affine transport* law based on the *affine covariant derivative* constructed from the pair $(\{\cdot\}_g, {}^{\hat{t}}K)$. If $\hat{\pi}$ is the affine 4-momentum of the charged particle, then we say $\hat{\pi}$ is *affinely parallel* along the trajectory iff $\hat{D}\hat{\pi}/Ds = 0$, where $\hat{D}/Ds$ is the affine directional covariant derivative along the path. Written in the zero translational gauge, this definition becomes

$$\left(\frac{\hat{D}\hat{\pi}}{Ds}\right)^j = \frac{Du^j}{Ds} + \varepsilon({}^{\hat{0}}K^j{}_k)u^k = 0 \tag{2.6}$$

where D/Ds is the linear directional covariant derivative and ε is the charge-to-mass ratio of the particle. In order to be compatible with Riemannian geometry, it can be shown (Norris, 1985), using the fact that $(d/ds)[\vec{u} \cdot \vec{u}] = 0$, that ${}^{\hat{0}}K_{(ij)} = 0$. Consequently, if we identify ${}^{\hat{0}}K$ with the negative of the electromagnetic field strength tensor, we obtain the equation

$$\frac{Du^j}{Ds} - \varepsilon F^j{}_k u^k = 0$$

Thus, in the $P(4)$ theory the Lorentz force law arises as an *affine 4-momentum geodesic*.

Based on the above identification, we may write the Einstein–Maxwell equations in terms of $P(4)$ quantities as (Norris, 1985)

$${}^{\hat{0}}\Phi_i = 0 \tag{2.7}$$

$${}^{\hat{0}}\Phi_{[ijk]} = 0 \tag{2.8}$$

$$R_{ij} - \tfrac{1}{2} g_{ij} R = {}^{\hat{0}}K_{ik}\,{}^{\hat{0}}K_j^{\ k} - \tfrac{1}{4} g_{ij}\,{}^{\hat{0}}K_{mn}\,{}^{\hat{0}}K^{mn} \tag{2.9}$$

The question that we address here is the following: can these equations be obtained from a $P(4)$ variational principle? The difficulty that one encounters in constructing a Lagrangian for this theory is that the unusual inhomogeneous gauge transformation laws of the affine quantities necessitate special care in the construction of such a Lagrangian, such that it is translationally gauge invariant. Our approach will also resolve the apparent problem that the left-hand side of equation (2.9) is translationally invariant, while the right-hand side appears not to be invariant.

3. THE PALATINI VARIATION

The variational technique that we employ in this paper will be of the *constrained Palatini type*. Exactly what we mean by this and our reasons for using this approach will be clarified by an examination of the more customary variational techniques, the Hilbert variation and the traditional or unconstrained Palatini variation (Lanczos, 1957; Ray, 1974; Safko and Elston, 1976).

We recall that historically these variational techniques were first applied to the Einstein Lagrangian of general relativity

$$L=(-g)^{1/2}g^{ij}R_{ij} \tag{3.1}$$

In the Hilbert variation, it is assumed that the linear connection is Riemannian and that the Lagrangian is a function of the metric tensor and its derivatives only. The variation then yields

$$\delta L=(-g)^{1/2}(R_{ij}-\tfrac{1}{2}g_{ij}R)\,\delta g^{ij} \tag{3.2}$$

It is noteworthy that all second derivatives of the metric can be eliminated in the Einstein Lagrangian by adding a total four-divergence to the Lagrangian. Furthermore, the Einstein Lagrangian is unique in the sense that it is the only Lagrangian which can be constructed of the 14 independent curvature invariants which leads to field equations of first or second order (Lanczos, 1957; Safko and Elston, 1976).

In the traditional version of the Palatini variation, on the other hand, the metric tensor and the linear connection are considered to be independent variables, and the linear connection is assumed to be symmetric but is otherwise arbitrary. In this case the variation yields

$$\delta L=(-g)^{1/2}[(R_{ij}-\tfrac{1}{2}g_{ij}R)\,\delta g^{ij}+(g^{ij}{}_{;k}+\tfrac{1}{2}\delta_k^j g^{im}{}_{;m}+\tfrac{1}{2}\delta_k^i g^{jm}{}_{;m})\,\delta\Gamma_{ij}{}^{k}] \tag{3.3}$$

and the variation of the linear connection leads easily to

$$g^{ij}{}_{;k}=0 \tag{3.4}$$

Thus, the variation picks the Riemannian connection out of the entire class of symmetric connections. It is well known that no other Lagrangian for general relativity has this property (Lanczos, 1957).

An alternative view of the Palatini variation has been offered by Lanczos (1957), Ray (1974), and Safko and Elston (1976). In their view, the Palatini variation of the Einstein Lagrangian is a special case of a constrained Lagrangian in which the Lagrange multiplier is found to be identically zero.

Specifically, the usual Einstein Lagrangian is replaced by

$$L=(-g)^{1/2}[g^{ij}R_{ij}+P_k{}^{ij}(\Gamma_{ij}{}^k-\{{}^{k}_{ij}\})] \tag{3.5}$$

where the $P_k{}^{ij}$ are Lagrange multipliers, $\Gamma_{ij}{}^k$ are the components of an arbitrary symmetric connection, and $\{{}^{k}_{ij}\}$ are the Christoffel symbols, which are understood to be expressed explicitly as a function of the metric tensor and its derivatives. Note that the constraint term is invariant under coordinate transformations because it involves the difference between two linear connections, which is a tensor. Variation with respect to the linear connection now gives

$$\frac{\delta L}{\delta\Gamma_{ij}{}^k}=(-g)^{1/2}(g^{ij}{}_{;k}+\tfrac{1}{2}\delta_k{}^j g^{im}{}_{;m}+\tfrac{1}{2}\delta_k{}^i g^{jm}{}_{;m}+P_k{}^{ij})=0 \tag{3.6}$$

Since the constraint term implies that $g^{ij}{}_{;k}=0$, this equation reduces to $P_k{}^{ij}=0$.

What has been gained by this alternative point of view? For one thing, one is now able to generalize the Palatini technique to Lagrangians for other field theories (Atkins *et al.*, 1977). For example, consider the Lagrangian in Minkowski space, $L=F_{ij}F^{ij}$, where F_{ij} is considered to be an arbitrary skew-symmetric tensor. Suppose we vary this Lagrangian with respect to F. We obtain nothing from the variation, since

$$\delta L/\delta F_{ij}=2F^{ij}=0 \tag{3.7}$$

However, if we constrain F to be the curl of a vector A by adding a constraint term to the Lagrangian, i.e.,

$$L=F_{ij}F^{ij}+H^{ij}(F_{ij}+A_{i,j}-A_{j,i}) \tag{3.8}$$

and treat F_{ij}, A_i, and H^{ij} as independent variables, we obtain the following equations:

$$\frac{\delta L}{\delta F_{ij}}=2F^{ij}+H^{ij}=0 \tag{3.9}$$

$$\frac{\delta L}{\delta H^{ij}}=F_{ij}+A_{i,j}-A_{j,i}=0 \tag{3.10}$$

$$\frac{\delta L}{\delta A_i}=-2H^{[ij]}{}_{,j}=0 \tag{3.11}$$

Equations (3.9) and (3.11) together imply that

$$F^{ij}{}_{,j}=0 \tag{3.12}$$

while equation (3.10) implies that

$$F_{[ij,k]}=0 \tag{3.13}$$

We see then, from equation (3.12), that one-half of Maxwell's equations arises as a *generalized force of constraint*. In this formulation, of course, we could have obtained the same field equations by building the constraint into the Lagrangian, but by doing so we lose the possibility of a deeper view into the structure of the theory. This is completely analogous to the fact that when constraints are built into the Lagrangians of simple mechanical systems, in general some information (i.e., the forces of constraint) is lost in the process. In the case of the Einstein Lagrangian, the result $P^{ij}_{k}=0$ may be interpreted as a statement that the generalized forces of constraint are zero (Safko and Elston, 1976).

Before continuing on to develop the $P(4)$ Lagrangian, we note that one can derive the Einstein Maxwell equations from a doubly constrained Palatini variation in the following manner. Let

$$L=(-g)^{1/2}[g^{ij}R_{ij}+\tfrac{1}{2}F_{ij}F^{ij}+P^{ij}_{k}(\Gamma_{ij}{}^{k}-\{{}^{k}_{ij}\})+H^{ij}(F_{ij}+A_{i,j}-A_{j,i})] \tag{3.14}$$

where g^{ij}, $\Gamma_{ij}{}^{k}$, F_{ij}, A_i, $P^{ij}{}_{k}$, and H^{ij} are all considered as independent variables. Since the additional terms do not involve covariant derivatives, the variation with respect to the symmetric connection $\Gamma_{ij}{}^{k}$ still results in the equation $P_k{}^{ij}=0$, which means that the constraint on the linear connection may be omitted. Meanwhile, the variations with respect to F_{ij}, H^{ij}, and A_i give equations (3.9)–(3.11) as before, while the variation with respect to g^{ij} yields the Einstein equation

$$\frac{\delta L}{\delta g^{ij}}=(-g)^{1/2}(R_{ij}-\tfrac{1}{2}g_{ij}R-F_{ik}F_j^k+\tfrac{1}{4}g_{ij}F_{mn}F^{mn})=0 \tag{3.15}$$

4. THE $P(4)$ LAGRANGIAN

In previous investigations (Norris, 1985, 1991; Kheyfets and Norris, 1988) it has been observed that a strong structural similarity exists between the theory of general relativity and the $\mathbb{R}^{4*}$ gauge theory of electromagnetism. This suggests that the vector potential plays a role in the $\mathbb{R}^{4*}$ theory similar to the role the metric plays in general relativity. In particular, it has been noted that there exists a special $\mathbb{R}^{4*}$ connection, which in the $\hat{0}$ translational gauge may be written as ${}^{0}K=-F$, and that this connection is constructed from derivatives of the vector potential in a manner analogous to the way that the special connection $\Gamma=\{\cdot\}_g$ in general relativity is built from derivatives of the metric. This suggests that in Minkowski spacetime the $\mathbb{R}^{4*}$ Lagrangian which is most closely analogous to the Einstein Lagrangian (3.1) is $L=A^i\,{}^{\hat{0}}\Phi_i$, since this is the "metric-like" quantity summed on a contraction of the $\mathbb{R}^{4*}$ curvature. Note from equation (2.5) that in Minkowski space this Lagrangian is $\mathbb{R}^{4*}$ invariant, since $R_{ij}=0$. Furthermore, we now show that, to within a total divergence, it is proportional to the standard electromagnetic Lagrangian. From equation (2.1) we find that

$$ {}^{\hat{0}}\Phi_i=-g^{jk}\,{}^{\hat{0}}K_{ki;j} \tag{4.1}$$

where we have used the fact that ${}^{\hat{0}}K_{(ij)}=0$. Consequently,

$$\begin{aligned}(-g)^{1/2}L_2&=(-g)^{1/2}(-A^i g^{jk}\,{}^{\hat{0}}K_{ki;j})\\ &=-((-g)^{1/2}A^i g^{jk}\,{}^{\hat{0}}K_{ki})_{,j}+(-g)^{1/2}A^{i;k}\,{}^{\hat{0}}K_{ki}\end{aligned}$$

Since ${}^{\hat{0}}K_{(ki)}=0$, the second term may be written as $-\frac{1}{2}(-g)^{1/2}F^{ik}F_{ik}$, and hence we have proved our assertion.

We now generalize this Lagrangian to curved spacetimes. We propose the Lagrangian

$$L=(-g)^{1/2}(L_1+L_2+L_3+L_4) \tag{4.2}$$

where

$$L_1=g^{ij}R_{ij}(\Gamma) \tag{4.3}$$

$$L_2=A^i({}^{\hat{t}}\Phi_i-{}^{\hat{t}}\tilde{\Phi}_i) \tag{4.4}$$

$$L_3=H^{ij}(P_{ij}+A_{j,i}-A_{i,j}) \tag{4.5}$$

$$L_4=L^{ij}({}^{\hat{t}}\tilde{K}_{ij}-t_{i;j}) \tag{4.6}$$

where H^{ij} and L^{ij} are sets of Lagrange multipliers. Here, we use the definitions

$$\vec{t}\overset{\text{def}}{=}\delta(\hat{0},\hat{t}) \tag{4.7}$$

and

$$P_{ij}\overset{\text{def}}{=}{}^{\hat{t}}K_{ij}-{}^{\hat{t}}\tilde{K}_{ij}\equiv{}^{\hat{0}}K_{ij} \tag{4.8}$$

Before proceeding, we make the following comments concerning the $P(4)$ Lagrangian and the variables from which it is constructed:

1. Regarding the physical interpretation of equation (4.7), recall that in Section 2 we identified the $\hat{0}$ translational gauge as the gauge in which the observed 4-momentum of a charged particle moving in a nonzero electromagnetic field is instantaneously the same as that of a comoving, freely falling, uncharged particle. By means of equation (4.7), we have modeled

the 4-momentum affine frame $(p, e_i(p), \hat{t}(p))$ at $p \in M$, with the triple $(p, e_i(p), \vec{t}(p))$, where the cotangent vector field $\vec{t}$ is defined as above (Norris, 1985; Kheyfets and Norris, 1988).

2. The $\mathbb{R}^{4*}$ connection ${}^{\hat{t}}\tilde{K}_{ij}$ is a flat $\mathbb{R}^{4*}$ connection in the sense that there exists a translational gauge in which the connection is identically zero. In this case, that particular gauge is the $\hat{0}$ gauge, as can be seen by examination of the constraint term L_4 and equation (2.2). The contracted $\mathbb{R}^{4*}$ curvature ${}^{\hat{t}}\tilde{\Phi}_i$ is constructed from ${}^{\hat{t}}\tilde{K}_{ij}$ and is therefore also flat. Note, from equations (2.1) and (2.5), that ${}^{\hat{0}}\tilde{\Phi}_i \equiv 0$. Physically, it appears that the flat $\mathbb{R}^{4*}$ connection ${}^{\hat{t}}\tilde{K}$ is related to the motion of the observer. In this case, as a consequence of our definition of the zero translation gauge, the observer is inertial. On the other hand, by demanding that ${}^{\hat{u}}\tilde{K} \equiv 0$ instead (and consequently, ${}^{\hat{0}}\tilde{K}_{ij} = -u_{j;i}$), where $\hat{u} = \hat{0} \oplus \vec{u}$ and $\vec{u}$ is the 4-velocity of a field of noninertial observers, preliminary results indicate that one may be able to obtain the equations of motion from the point of view of a noninertial observer. We shall return to this point in future publications.

3. The total Lagrangian is translationally invariant, as can be seen by the examination of the individual terms. L_1 contains no quantities with affine transformation laws, hence it is invariant under translations. L_2 and L_3 involve the difference between contracted affine curvatures and affine connections, respectively, and consequently the inhomogeneous transformation terms are canceled out in both L_2 and L_3. One can show L_4 to be invariant by examination of equations (2.2) and (4.7). Also note that the difference tensor P_{ij} is translationally invariant.

4. The covector field A_i is not to be thought of, at this point in our development, as the electromagnetic vector potential. Rather, it is simply a covector field on M which is coupled linearly to ${}^{\hat{t}}\Phi_i$ and as such, it does not appear to have the classical gauge freedom embodied by the transformation $A_i \to \bar{A}_i = A_i + f_{,i}$. We shall later show that due to the antisymmetry of the difference tensor P_{ij}, A_i does acquire this gauge freedom.

5. As before, the linear connection Γ^k_{ij} in L_1 is symmetric but otherwise arbitrary. We have not, however, included terms in the total Lagrangian which constrain the connection to be metric, as we did in equations (3.5) and (3.14), since the inclusion of such terms again results in the Lagrange multipliers being identically zero.

Before proceeding with the variation, we shall remove a total 4-divergence from the term L_2, as we did above in the case of Minkowski spacetime. Writing ${}^{\hat{t}}\Phi_i$ and ${}^{\hat{t}}\tilde{\Phi}_i$ in terms of covariant derivatives of ${}^{\hat{t}}K_{ij}$ and ${}^{\hat{t}}\tilde{K}_{ij}$, respectively, we find that

$$L_2 = (\text{4-divergence}) + Q^{ij} P_{ji} \tag{4.9}$$

where

$$Q^{ij} \stackrel{\text{def}}{=} A^{i;j} - g^{ij} A^k{}_{;k} \tag{4.10}$$

Therefore, we shall replace L_2 by

$$L_2' = Q^{ji} P_{ij} \tag{4.11}$$

in the total Lagrangian.

Since $P_{(ij)} = 0$, it is clear that only the antisymmetric part of Q^{ij} enters into the contraction. L_2', therefore, reduces to the usual electromagnetic Lagrangian, quadratic in the field strength, and it is therefore invariant to classical gauge transformations. Note that we have not destroyed translational invariance in the process because, as before, the tensor $P_{ji} = {}^{t}K_{ji} - {}^{t}\tilde{K}_{ji}$ is the difference between two $\mathbb{R}^{4*}$ connections.

We now vary the Lagrangian $L = (-g)^{1/2}(L_1 + L_2' + L_3 + L_4)$ with respect to the variables g_{ij}, Γ^k_{ij}, A_i, ${}^{t}K_{ij}$, H^{ij}, and L^{ij} and their derivatives. We obtain as a result the following equations:

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta g^{ij}} = R_{ij} - \tfrac{1}{2} g_{ij} R + P_{ik} A^k{}_{;j} + P_{ki} A_j{}^{;k} + P_k{}^k A_{(i;j)}$$
$$+ P_{(ji)} A^k{}_{;k} - \tfrac{1}{2} g_{ij} Q^{km} P_{mk} = 0 \tag{4.12}$$

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta A_i} = -P^{ji}{}_{;j} - g^{ik} P^j{}_{j;k} - g^{ik}{}_{;k} P_j{}^j - 2H^{[ji]}{}_{;j} = 0 \tag{4.13}$$

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta\, {}^{t}K_{ij}} = Q^{ji} + H^{ij} = 0 \tag{4.14}$$

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta H^{ij}} = P_{ij} + A_{j,i} - A_{i,j} = 0 \tag{4.15}$$

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta L^{ij}} = {}^{t}\tilde{K}_{ij} - t_{i;j} = 0 \tag{4.16}$$

$$\frac{1}{(-g)^{1/2}} \frac{\delta L}{\delta \Gamma_{ij}^{\;k}} = g^{ij}{}_{;k} - \delta_k^{(j} g^{i)m}{}_{;m} + P^{(ij)} A_k$$
$$+ P_m{}^m g^{ij} A_k + Q^{(ji)} t_k + H^{(ij)} t_k = 0 \tag{4.17}$$

From equation (4.15) we obtain the identification of $P = {}^{\hat{0}}K$ as the curl of A and thereby the antisymmetry of P. When this and the definition of H^{ij} obtained from equation (4.14) are combined with equation (4.9) we obtain

$$R_{ij} - \tfrac{1}{2} g_{ij} R = P_{ik} P_j^{\ k} - \tfrac{1}{4} g_{ij} P_{mn} P^{mn} = {}^{\hat{0}}K_{ij}\, {}^{\hat{0}}K_j^{\ k} - \tfrac{1}{4} g_{ij}\, {}^{\hat{0}}K_{mn}\, {}^{\hat{0}}K^{mn} \tag{4.18}$$

The definition of H^{ij} along the antisymmetry of P_{ij}, when combined with equations (4.13) and (4.17), gives

$$P^{ij}{}_{;j} = {}^{\hat{0}}K^{ij}{}_{;j} = 0 \tag{4.19}$$

and

$$g^{ij}{}_{;k} = 0 \tag{4.20}$$

Equations (4.18) and (4.19) are the geometrical, source-free Einstein–Maxwell affine field equations, as derived from a $P(4)$ variational principle. Moreover, from equation (4.20) we see that the linear connection is Riemannian.

5. EXTENSION OF THE *P*(4) LAGRANGIAN TO INCLUDE SOURCES

The $P(4)$ theory can be extended easily to accommodate sources by the inclusion of interaction terms $(-g)^{1/2} A^i J_i$ and $-(-g)^{1/2} g^{ij} T_{ij}$ in the Lagrangian. When this is done we obtain the following modifications of equations (4.18) and (4.19):

$$P^{ij}{}_{;j} = -J^i \tag{5.1}$$

and

$$R_{ij} - \tfrac{1}{2} g_{ij} R = P_{ik} P_j^{\ k} - \tfrac{1}{4} g_{ij} P_{mn} P^{mn} + T_{ij} \tag{5.2}$$

In fact, the Lagrangian can also be modified to include magnetic charges as well. This is accomplished by the introduction of a second vector potential B^i (Cabibbo and Ferrari, 1962). In Minkowski spacetime, we modify L_2 and L_3 as follows:

$$L_2 = A^i\, {}^{\hat{0}}\Phi_i + \tfrac{1}{2} B^i\, {}^{\hat{0}}\Phi_i^* \tag{5.3}$$

and

$$L_3 = H^{ij}(P_{ij} + A_{j,i} - A_{i,j} + \varepsilon_{ijkl} B^{k,l}) \tag{5.4}$$

where we define

$${}^{\hat{\imath}}\Phi_i^* \overset{\text{def}}{=} {}^{\hat{\imath}}\Phi_{ij}^{*j} \tag{5.5}$$

In addition, we may include an interaction term $-B^i\tilde{J}_i$, where $\tilde{J}_i$ represents the magnetic current density. As a result, we obtain the additional field equation

$$(B^{j,i}-B^{i,j})_{,j}=\overset{*}{P}{}^{ij}{}_{,j}=\tilde{J}^i \tag{5.6}$$

Had the last interaction term been omitted, we would obtain instead the $U(1)$ Bianchi identity in terms of the $\mathbb{R}^{4*}$ curvature, that is, ${}^{\hat{0}}\Phi_i^*=0$. The inclusion of these terms in the Lagrangian does not create any difficulties when we generalize to curved spacetime, since the introduction of the magnetic potential B^i does not affect the compatibility of the $\mathbb{R}^{4*}$ connection with Riemannian geometry. Recall that this compatibility condition, as stated in Section 2, was ${}^{\hat{0}}K_{(ij)}=0$. We subsequently assumed that ${}^{\hat{0}}K$ was minus the curl of A, whereas we might have chosen a more general form for ${}^{\hat{0}}K$. It has been shown (Kobe, 1983) that any antisymmetric, second-rank tensor field which vanishes at spatial infinity can always be written as the curl of one vector field plus the dual of the curl of a second vector field. Consequently, one may observe that the modification of L_3 in equation (5.4) is a natural choice. In addition, it is worth noting that the generalization to curved spacetime does not necessitate the inclusion of a difference term to accompany $\frac{1}{2}B^i\,{}^{\hat{t}}\Phi_i^*$ [as was the case with $L_2=A^i({}^{\hat{t}}\Phi_i-{}^{\hat{t}}\tilde{\Phi}_i)$], since ${}^{\hat{t}}\Phi_i^*$ is already translationally invariant.

6. CONSERVATION LAWS ARISING FROM THE INVARIANCE PROPERTIES OF THE $P(4)$ LAGRANGIAN

As we noted in Section 4, our final Lagrangian is invariant under translational gauge transformations. However, there are no conservation laws which occur as a consequence of this invariance property. This is due to the fact that an infinitesimal translational gauge transformation $\hat{t}\to\hat{t}\oplus\varepsilon\vec{s}$, where ε is an infinitesimal, leaves the difference tensor P_{ij} invariant. On the other hand, an infinitesimal gauge transformation of the A^i leads to positive results. Once the 4-divergence is removed from the term L_2, it is clear that the covector field A^i has the gauge freedom embodied by the transformation $A^i\to\bar{A}^i=A^i+f^{,i}$. However, as we shall now show, the L_2 term is classically gauge invariant up to a 4-divergence, provided the difference tensor P_{ij} is antisymmetric. Consider the term introduced in L_2 by the addition of the gradient $f^{,i}$ to the vector field A^i, that is,

$$\begin{aligned}
&(-g)^{1/2}f^{,i}g^{jk}(P_{kj;i}-P_{ki;j})\\
&=(-g)^{1/2}(f^{,i}g^{jk}-f^{,j}g^{ik})P_{kj;i}\\
&=[(-g)^{1/2}(f^{,i}g^{jk}-f^{,j}g^{ik})P_{kj}]_{,i}+(-g)^{1/2}(f^{,j;k}-g^{jk}f^{,i}{}_{;i})P_{kj}
\end{aligned}$$

A sufficient condition for the vanishing of the second term is that $P_{(kj)}=0$, which is weaker than the condition imposed by the constraint term L_3.

The variation of the vector A^i, which may now be considered as a vector potential, due to an infinitesimal gauge transformation $A^i \to \bar{A}^i = A^i + \varepsilon f^{,i}$ is given by

$$\delta^*(A^i) \stackrel{\text{def}}{=} \bar{A}^i - A^i = \varepsilon f^{,i} \tag{6.1}$$

Since L_1, L_3, and L_4 are invariant under this variation we obtain

$$\begin{aligned} 0 \equiv \delta^*((-g)^{1/2}L) &= (-g)^{1/2}\, {}^{\hat{0}}\Phi_i \delta^* A^i = (-g)^{1/2}\, {}^{\hat{0}}\Phi_i(\varepsilon f^{,i}) \\ &= (\varepsilon(-g)^{1/2}\, {}^{\hat{0}}\Phi_i f)^{,i} - \varepsilon(-g)^{1/2}\, {}^{\hat{0}}\Phi_i{}^{;i} f \end{aligned}$$

Since f is an arbitrary scalar field, it follows that

$${}^{\hat{0}}\Phi_i{}^{;i} = 0 \tag{6.2}$$

which is equivalent to $F_{ij}{}^{;ij}=0$. Note that since this result does not depend on the fact that ${}^{\hat{0}}\Phi_i = 0$, this may be considered as a "strong" conservation law. Similarly, an infinitesimal gauge transformation of the magnetic vector potential B^i leads to a conservation law of the form ${}^{\hat{t}}\Phi_i^{*;i}=0$, which is equivalent to $F_{ij}^{*;ij}=0$. We have obtained, therefore, conservation of both electric and magnetic charge.

7. CONCLUSION

The $P(4)$ theory provides a unification of gravitation and electromagnetism at the classical level through the identification of the electromagnetic field strength tensor with the $\mathbb{R}^{4*}$ component of a $P(4) = O(1,3) \otimes \mathbb{R}^{4*}$ generalized *affine* connection on spacetime. This is a marked departure from the usual approach to unification, which ordinarily seeks to include electromagnetism within general relativity by making the field strength tensor a part of some generalized *linear* connection.

The fundamentally new idea in the $P(4)$ theory is to model the instantaneous 4-momentum spaces of a charged particle moving through an electromagnetic field as four-dimensional affine spaces. The operational meaning of the Lorentz force law, namely that the force experienced by a charged particle is to be measured relative to an instantaneously comoving inertial observer, leads to the identification (Norris, 1985) of a particular gauge, the $\hat{0}$ gauge, in which the $\mathbb{R}^{4*}$ component of the full $P(4)$ connection is antisymmetric. The subsequent identification of the $\mathbb{R}^{4*}$ component in this gauge with the negative of the electromagnetic field strength tensor leads to the realization of the Lorentz force law as an affine 4-momentum geodesic. One may then obtain from this identification the Einstein–Maxwell affine field equations in terms of contractions of the $O(1,3)$ and $\mathbb{R}^{4*}$ curvature components.

We believe that the $P(4)$ theory has a number of advantages over other attempts at the classical unification of gravity and electromagnetism, among which are the following. First, the $P(4)$ theory is simpler and more fundamental in terms of the underlying physical concepts. Second, the $O(1,3)$ and $\mathbb{R}^{4*}$ components of the $P(4)$ connection are placed on equal footing in the $P(4)$ theory. This is in contrast to the so called "already unified" theory of Rainich, Misner, and Wheeler (RMW), in which the field strength tensor is reduced to the role of the "Maxwell square root" of the Ricci tensor (Rainich, 1925; Misner and Wheeler, 1957). Third, in the $P(4)$ theory, one is able to obtain the Lorentz force law as an affine 4-momentum geodesic in a simple and natural manner. Finally, there is a remarkably close structural similarity between the $\mathbb{R}^{4*}$ theory of electromagnetism and general relativity.

In this paper we have sought to complete one aspect of the $P(4)$ theory by the introduction of a Lagrangian from which the Einstein–Maxwell affine field equations may be derived. In addition we have shown that these field equations transform covariantly under $P(4)=O(1,3)\otimes\mathbb{R}^{4*}$ transformations. The discovery of this variational principle is a further advantage of the $P(4)$ theory over the RMW theory, since no satisfactory Lagrangian has been uncovered for the RMW theory.

It is significant that the electromagnetic sector of our Lagrangian (L_2), which arises from a straightforward generalization of the Einstein Lagrangian, turns out to be, to within a four-divergence, the usual Lagrangian of electromagnetism, quadratic in the field strength. This, it would appear, is another example of the successful exploitation of the structural similarities between $\mathbb{R}^{4*}$ electromagnetism and general relativity mentioned above. As we have demonstrated in Section 5, the $P(4)$ Lagrangian can be easily extended to include matter currents and both electric and magnetic charges. The latter was achieved by the introduction of a magnetic vector potential which we have shown is completely consistent with $P(4)$ geometry. Here it should be stressed that the $\mathbb{R}^{4*}$ component of the $P(4)$ connection can only be thought of as constructed solely from these two potentials in the $\hat{0}$ translational gauge. Both of these potentials acquire the classical gauge freedom of the vector potential and we have used the invariance of the total Lagrangian under classical gauge transformations of these two potentials to derive conservation laws for both electric and magnetic charge.

An interesting issue which has arisen in this paper is the physical significance of the flat $\mathbb{R}^{4*}$ connection $\tilde{K}$. As we have mentioned above, we demand that ${}^{\hat{0}}\tilde{K}_{ij}\equiv 0$. Physically, this means that at every point along the trajectory of a charged particle, we choose to measure the Lorentz forced with respect to an instantaneously comoving inertial observer without any "noninertial affine effects" arising due to the flat connection $\tilde{K}$. It appears that an alternative choice of observers may be implemented by a different demand on the flat connection $\tilde{K}$ as we mentioned in Section 4. If, as we have speculated, the flat connection is related to the motion of observers along the trajectory of the charged particle, then one could say that it is neither the connection of the pure electromagnetic field K nor the connection of the observer $\tilde{K}$ that is physically observable, but that it is the difference between these connections that has physical relevance. Perhaps we may then sum up the affine nature of spacetime in the following way: the motion of a particle in spacetime is not determinable apart from the motion of the observer of the particle and therefore the very act of observation must involve the difference between affine objects. This appears to be the case for at least three different classes of affine objects. The first of these are the points of spacetime themselves. If points p and q are elements in some local coordinate patch (U, x) of spacetimes, then the coordinate maps $x^i: U \to \mathbb{R}^4$ can be used to construct a difference function $\delta_0: U \times U \to \mathbb{R}^4$ defined by $\delta_0(p, q) = (x^i(p) - x^i(q))$, thus making $(U, \mathbb{R}^4, \delta_0)$ into an affine space. This is the usual "relativity of events" in relativistic physics. In this sense, the relative displacement of a particle at point p and an observer at point q is the affine difference between p and q, namely $\delta_0(p, q)$. In the $P(4)$ theory, these ideas have been extended to the 4-momentum spaces of charged particles to define a "relativity of 4-momentum" for classical charged particles. In this case, the observed 4-momentum of charged particles is the affine difference between the affine 4-momentum of the particle and the local zero of 4-momentum, the latter of which is related to the motion of observers. Finally, in our variational principle we have seen the electromagnetic field strength tensor appear as the difference between two $\mathbb{R}^{4*}$ affine connections, one of which appears to be related to the observer. This may also be considered as an affine difference, the range of the difference function being the space of second-rank tensors on spacetime.

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